Abstract varieties

We know about affine varieties and projective varieties, but now we want to define varieties more generally. One way to do this is to take the product of projective spaces and affine spaces and take an "algebraic set" in there.

We know that $\mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$, so any algebraic set in the product of affine spaces will again be affine.

However, $\mathbb{P}^n \times \mathbb{P}^m \not\cong \mathbb{P}^{n+m}$, so we need a new method for defining algebraic sets.

Multiprojective space

Def: Let $k[X,Y] = k[x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}]$. $F \in k[X,Y]$ is a <u>biform of degree</u> (p,q) is a form of degree p (resp. q) in the x_i (resp y_i).

Ex: $x_1 y_1 y_2^3 + x_2 y_3^4$ is a biform of degree $(1,4)$ in $k[x_1, x_2, y_1, y_2, y_3]$.

Every $F \in k[X,Y]$ can be written uniquely as

$$F = \sum_{p,q} F_{p,q}$$

where $F_{p,q}$ is a biform of degree (p,q).

We can now define an algebraic set in $\mathbb{P}^n \times \mathbb{P}^m$ to be the set of zeros of a set of biforms S. i.e.

$$V(S) = \{(X,Y) \in \mathbb{P}^n \times \mathbb{P}^m \mid F(X,Y) = 0 \quad \forall F \in S\}.$$

We can analogously define the ideal of a subset of $\mathbb{P}^n \times \mathbb{P}^m$:
If $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$, $I(V) = \{f \in k[X,Y] \mid f(X,Y) = 0 \;\; \forall \;\; (X,Y) \in V\}$.

And the bihomogeneous coordinate ring: $\Gamma_b(V) = k[X,Y]/I(V)$

And the field of rational functions: $k(V) = \left\{ F/G \;\middle|\; \begin{matrix} F, G \text{ biforms of the same} \\ \text{bidegree in } \Gamma(V) \end{matrix} \right\}$
w/ local rings defined as before.

We can extend all these definitions to finite products of projective spaces, and in fact, products of projective spaces and affine space:

$$\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r} \times \mathbb{A}^m.$$

In this case, a polynomial should be homogeneous in each of the variables corresponding The the projective spaces, but there is no restriction on those corresponding to the affine space.

Ex: The map $f: \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^N$ $(N = (n+1)(m+1) - 1)$ given

$$([x_0 : \dots : x_n], [y_0 : \dots : y_m]) \longmapsto [x_0 y_0 : x_0 y_1 : \dots : x_n y_m]$$

is called the Segre embedding. We'll soon see that this is a morphism, and in fact it's isomorphic onto its image, which is closed, i.e. $\mathbb{P}^n \times \mathbb{P}^m$ is itself a projective variety.

Varieties in general

We can give $X = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$ the Zariski topology in the same way as in affine/projective space:

$U \subseteq X$ is open $\iff$ $X \setminus U$ is an algebraic set.

Any subset $Y \subseteq X$ inherits the Zariski topology: the open sets of Y are of the form $Y \cap U$ where $U \subseteq X$ is open.

Def: Let V be a nonempty irreducible algebraic set in $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$. Any open subset $X \subseteq V$ is called a variety.

$k(X) := k(V)$ is the field of rational functions on X, and $\mathcal{O}_P(X) := \mathcal{O}_P(V)$ the local ring of X at P.

If X is a variety, and $U \subseteq X$ open, $f \in k(X)$ is regular on U if f is defined at each $P \in U$.

The ring of regular functions on U is

$$\Gamma(U, \mathcal{O}_X) := \{ f \in k(X) \mid f \text{ is regular on } U \}$$

That is, $\Gamma(U, \mathcal{O}_X) = \bigcap_{P \in U} \mathcal{O}_P(X)$.

Note: If $U = X$ is an affine variety, $\Gamma(X, \mathcal{O}_X) = \Gamma(X)$, the coordinate ring of X.

Ex: If $X = \mathbb{P}^n$, $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \left\{ \frac{F}{G} \;\middle|\; \begin{array}{l} F, G \text{ homog.} \\ \text{of same deg} \end{array},\ G(P) \neq 0 \ \forall\, P \in \mathbb{P}^n \right\}$

$$= k \neq \Gamma_h(\mathbb{P}^n)$$

In fact, $\Gamma(X, \mathcal{O}_X) = k$ for every projective variety X.

In general, each element $f \in \Gamma(U, \mathcal{O}_X)$ determines a function $U \to k$. In fact, each distinct element determines a unique function. That is:

Claim: The natural ring map

$$\Gamma(U, \mathcal{O}_X) \to \mathcal{F}(U, k) = \{\text{functions } U \to k\}$$

is injective.

Pf: Let $\gamma \in \Gamma(U, \mathcal{O}_X)$ s.t. $\gamma(P) = 0 \ \forall\, P \in U$. We need to show $\gamma = 0$.

$X \subseteq \mathbb{P}^{n_1} \times \dots \times \mathbb{A}^m$. $k(X) = k(\bar{X})$, so we can assume X is closed.

For each projective space, choose an affine chart U_i

s.t. $U \cap \underbrace{U_1 \times U_2 \times \dots \times \mathbb{A}^m}_{\mathbb{A}^N} \neq \emptyset$.

set $U' := U \cap \mathbb{A}^n$ and $X' := X \cap \mathbb{A}^n$.

Then $X' \subseteq X$ is open, so $k(X') = k(X)$, and $X' \subseteq \mathbb{A}^n$ is closed, so it's an affine variety, and $U' \subseteq X'$ is open.

Moreover, $U' \subseteq U$, so $\Gamma(U, \mathcal{O}_X) \subseteq \Gamma(U', \mathcal{O}_{X'})$.

Thus, $\gamma = \frac{f}{g}$, $f, g \in \Gamma(X')$, $g(P) \neq 0$ $\forall$ $P \in U'$. Then $f(P) = 0$ $\forall$ $P \in U$, so $f = 0 \Rightarrow \gamma = 0$. $\square$

Def: If X is a variety, and $Y \subseteq X$ is closed, Y is <u>irreducible</u> if it's not the union of two proper closed subsets.

Note: We will mostly care about the case of projective varieties, or even open subsets of projective varieties, but understanding the general theory is important in order to discuss concepts like dimension, birationality, and resolution of singularities.

Morphisms of Varieties

Let X and Y be varieties. A <u>morphism</u> from X to Y is a function $\varphi : X \longrightarrow Y$ such that

1.) φ is continuous. ($\varphi^{-1}(U)$ is open $\forall$ $U \subseteq Y$ open)

2.) For every open set $U \subseteq Y$, if $f \in \Gamma(U, \mathcal{O}_Y)$, then

$\varphi^*(f) := f \circ \varphi$ is in $\Gamma(\varphi^{-1}(U), \mathcal{O}_X)$.

That is, the function $f \circ \varphi$ is regular on U.
(Note φ^* is a homomorphism.)

An <u>isomorphism</u> of X with Y is a morphism $X \to Y$ s.t. its inverse is a morphism.

<u>Remark</u>: This definition of morphism agrees w/ our definition in the case of affine/projective varieties.

<u>Remark</u>: If $U \subseteq X$ is open and $\varphi : X \to Y$ is a morphism, $\varphi|_U \to Y$ is also a morphism.

<u>Ex</u>: Consider the map $\varphi : \mathbb{A}^n \longrightarrow U_{n+1} \subseteq \mathbb{P}^n$ given
$$(x_1, \dots, x_n) \longmapsto [x_1 : \dots : x_n : 1]$$

Can check: If $W \subseteq U_{n+1}$ open, then $\varphi^{-1}(W)$ is open (it's the complement of the dehomogenization of $I(\mathbb{P}^n \setminus W)$)

If $\frac{F}{G}$ is defined on W, then $\frac{f}{g}$ defined on $\varphi^{-1}(W)$.
dehom

So φ is a morphism. By a similar argument, the inverse φ^{-1} is also a morphism, so, in fact $\mathbb{A}^n \cong U_{n+1} \cong U_i \;\; \forall i$.

<u>Ex</u>: Define $V = V(xy-1) \subseteq \mathbb{A}^2$. $\Gamma(V) = k[x,y]/(xy-1)$

Consider the morphism $V \longrightarrow \mathbb{A}^1$ defined

$$(x,y) \longmapsto x$$

The image is $W = \mathbb{A}^1 - \{0\}$, an open set, so define

$\varphi : V \longrightarrow W$ to be the corresponding morphism. I claim this is an isomorphism:

Consider the morphism $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$ defined

$$[a:b] \longmapsto [a^2 : b^2 : ab]$$

Check: the image is $V_p(xy - z^2)$.

Thus, if we restrict to $\mathbb{P}^1 - \{[0:1],[1:0]\} = \underset{\substack{\| \\ W}}{U_2 - \{0\}} \cong \mathbb{A}^1 - \{0\}$, and the image to V, we get

$$[a:1] \longmapsto [a^2 : 1 : a], \text{ or}$$

$$a \longmapsto \left(a, \tfrac{1}{a}\right)$$

which is a morphism!! Even though $\frac{1}{a}$ is not a polynomial on $\mathbb{A}^1$. Thus, $V \cong W$, so W is an affine variety, even though it is open in $\mathbb{A}^1$.

In fact, this can hold more generally:

Prop: Let V be an affine variety and $f \in \Gamma(V)$, $f \neq 0$. Let $V_f = \{P \in V \mid f(P) \neq 0\} = V - \Gamma(f)$.

1.) $\Gamma(V_f) = \Gamma(V)[1/f] = \{ a/f^n \in k(V) \mid a \in \Gamma(V), n \in \mathbb{Z} \}$

2.) V_f is an affine variety. i.e. $V_f \cong V'$, some $V' \subseteq \mathbb{A}^n$ closed.

Pf: 1.) from past HW.

2.) See Fulton. Idea: If $I = I(V) \subseteq k[x_1, \ldots, x_n]$, set $I' = (I, x_{n+1} f - 1)$.

Claim: $V' = V(I') \cong V_f$. $\square$

In fact: For any variety X, $P \in X$, and $P \in U \subseteq X$ open, $\exists$ $V \subseteq X$ open s.t. $P \in V$ and V is an affine variety. That is, X is covered by open sets that are affine varieties, called "affine opens".